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Stability of a viscous fluid in an oscillating gravitational field

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The stability of the interface of a viscous incompressible fluid superimposed on a massless fluid is studied for the case of an oscillating gravitational field. For the viscous case, the dispersion relation is shown to represent an infinite determinant of the Hill type, which is investigated analytically. The method presented allows one to find the whole dispersion curve of the instability and its asymptotics in an explicit form. The stabilizing effect of the externally imposed oscillations leads to the appearance of stability windows on the growth rate spectrum. Illustrations are given for the influence of all the parameters of the problem on this effect.

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I. INTRODUCTION

The Rayleigh-Taylor (RT) instability was originally discovered as the instability of the interface of two superimposed fluids in a gravitational field. Theoretical interest in this phenomenon today is related to its appearance in a number of physical objects both in space and laboratory fluids and plasmas [1]. The influence of various physical parameters and processes on the RT growth rate γ is often studied in the light of finding a stabilizing effect which leads to a reduction of the value of γ, at least at given wavelengths.

One can note a similarity between the linear growth of RT instability in incompressible fluids and the motion of an inverted pendulum (a conventional mechanical pendulum consisting of a long rigid stick with a small heavy body attached to its end). Indeed, the motion of an incompressible fluid over a massless incompressible medium in a gravity field consists of interchanging the opposite volume parts across the interface. Furthermore, the expressions for the characteristic frequencies of the RT (often called interchange) instability and the pendulum are alike, $\gamma=\sqrt{gk}$ and $\omega=\sqrt{g/l}$ (l is the length of the pendulum).

A well-known classic experiment for preventing an inverted pendulum from falling was described in Ref. [2]. The idea of the experiment was to make the point of attachment of the pendulum to vibrate with a rather small amplitude ξ_0, but with a high frequency Ω. The essential point here is to make the effective changing gravity field $g_{\mathrm{osc}}=\xi_0\Omega^2$ higher than the original one, g. Further developing the analogy between a pendulum of a fixed length and RT perturbations of a fluid of a given wavelength, we can expect an analogous effect in the case of two incompressible fluids.

The present paper analyzes the possibility of a reduction of the instability growth rate in the region of intermediate wavelengths for the case when the acceleration field is oscillating. To obtain a basic notion of the effect, a simple case of incompressible fluids is considered. The authors of Ref. [5] already derived a dispersion relation for this stability problem in the form of an infinite determinant. This determinant was further analyzed, mostly numerically, in order to find stability boundaries. As opposed to Ref. [5], here we propose an analytical method which allows one to obtain an explicit form of the dispersion relation and avoid a large amount of numerical work.

II. DISPERSION EQUATION FOR LINEAR INSTABILITY GROWTH IN VISCOUS FLUID

Figure 1 illustrates the statement of the problem. Two incompressible nonmixing fluids are superimposed, both of them harmonically oscillating in a direction perpendicular to the interface. In practice, this situation can by realized by putting two liquids in a cylinder with rigid walls, and making this cylinder vibrate in the direction of **g**. The corresponding experiment was described in Ref. [3]. In the chosen frame of reference the fluids are immobile as a whole, and the pressure instantaneously compensates for the change of the effective acceleration field. The axis Z is normal to the fluid surface, and the axes X and Y are in the plane of the fluid interface, so that $\mathbf{g}=ge_z$.

For growth rates of the surface perturbation γ, exceeding Ω, the forced oscillations effectively increase the initial gravity force g. Further analysis will be performed for the case when the instability boost at shorter wavelengths, corresponding to higher values of γ, is limited by the viscosity. Also, to avoid tedious calculations, we will examine the case of a heavy fluid superimposed over a massless one.

For a viscous incompressible fluid in a noninertial frame of reference

$$\frac{d\mathbf{v}}{dt}=-\frac{\nabla p}{\rho}+\mathbf{g}+\mathbf{g}_{\mathrm{osc}}+\frac{\eta}{\rho}\nabla\mathbf{v}, \tag{1}$$

$$\nabla\cdot\mathbf{v}=0, \tag{2}$$

where we choose $|\mathbf{g}_{\mathrm{osc}}|=\xi_0\Omega^2\cos(\Omega t)$.

The initial condition for Eqs. (1) and (2) corresponds to the equilibrium state $\mathbf{v}_0=0$, or

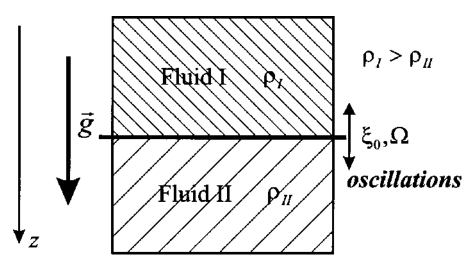

FIG. 1. General statement of the problem.

$$p_0=\rho[g+\xi_0\Omega^2\cos(\Omega t)]z+p_{\rm ext}. \tag{3}$$

The boundary conditions are formulated as follows [4]:

$$\left(-p+2\eta\frac{\partial \nu_z}{\partial z}\right)\bigg|_{\rm surface}=-p_{\rm ext}, \tag{4}$$

$$\left(\frac{\partial \nu_x}{\partial z}+\frac{\partial \nu_z}{\partial x}\right)\bigg|_{\rm surface}=0 \quad \left(\frac{\partial \nu_y}{\partial z}+\frac{\partial \nu_z}{\partial y}\right)\bigg|_{\rm surface}=0. \tag{5}$$

In a linear approximation Eqs. (1) and (2) yield $\boldsymbol{\nu}=\boldsymbol{\nu}_0+\boldsymbol{\nu}_1$ for small perturbations of the velocity, and $p=p_0+p_1$ for the pressure.

$$\frac{\partial \mathbf{v}_1}{\partial t}=-\frac{\boldsymbol{\nabla} p_1}{\rho}+\frac{\eta}{\rho}\Delta_1. \tag{6}$$

$$\nabla\cdot\mathbf{v}_1=0. \tag{7}$$

Taking the divergence of Eq. (6) and taking into account Eq. (7), we obtain

$$\Delta p_1=0. \tag{8}$$

Since the equilibrium system is homogeneous in the plane (x, y), we can make a Fourier transform over these variables. As a result we will have corresponding Fourier components with the wave vector $\mathbf{k}=(k_x,k_y)$. In order to simplify our calculations further, let us change the directions of the axes x and y, so that $\mathbf{k}=k\mathbf{e}_x$ and $k=|\mathbf{k}|$. In other words, we are thus eliminating the unnecessary variable y from our problem. Also, as we have externally imposed oscillations, let us look for the solutions of the perturbed system of equations (6) and (8) in the form of Floquet series over the time variable. Summarizing all the remarks made above, for the Fourier components we have

$$\mathbf{v}_1=e^{-i\omega t}\sum_{n=-\infty}^{\infty}e^{in\Omega t}\mathbf{v_{n,k}}(z)e^{ikx}, \tag{9}$$

$$p_1=e^{-i\omega t}\sum_{m=-\infty}^{\infty}e^{im\Omega t}p_{m,k}(z)e^{ikx}. \tag{10}$$

From Eq. (8), $p_{m,k}(z)=p_{m,k}e^{kz}(z<0)$, and for the velocity component ν_x we have, from Eq. (6),

$$(-i\omega+in\Omega)\nu_{x,n,k}(z)=-\frac{1}{\rho}ikp_{n,k}e^{kz}+\frac{\eta}{\rho}\left(-k^2\nu_{x,n,k}(z)+\frac{\partial^2\nu_{x,n,k}(z)}{\partial z^2}\right). \tag{11}$$

The general solutions of Eq. (11) are as follows:

$$\nu_{x,n,k}(z)=\nu_{x,n,k}e^{q_n z}+p_{n,k}\frac{k}{\rho}\frac{e^{kz}}{\omega-n\Omega}, \tag{12}$$

$$q_n=\left(k^2+i\rho\frac{(n\Omega-\omega)}{\eta}\right)^{1/2}, \quad \mathrm{Re}\,q_n>0. \tag{13}$$

Condition $\mathrm{Re}\,q_n>0$ is necessary for the perturbations to disappear at $z\rightarrow-\infty$.

The same calculations for ν_z result in

$$\nu_{z,n,k}(z)=\nu_{z,n,k}e^{q_n z}-ip_{n,k}\frac{k}{\rho}\frac{e^{kz}}{\omega-n\Omega}. \tag{14}$$

Thus all the expressions for the pressure and velocity components are

$$p_1=e^{-i\omega t}\sum_{m=-\infty}^{\infty}e^{im\Omega t}p_{m,k}e^{ikx+kz}, \tag{15}$$

$$\nu_{1x}=e^{-i\omega t}\sum_{n=-\infty}^{\infty}e^{in\Omega t}\left(\nu_{x,n,k}e^{ikx+q_n z}+p_{n,k}\frac{k}{\rho}\frac{1}{\omega-n\Omega}e^{ikx+kz}\right), \tag{16}$$

$$\nu_{1z}=e^{-i\omega t}\sum_{n=-\infty}^{\infty}e^{in\Omega t}\left(\nu_{z,n,k}e^{ikx+q_n z}-ip_{n,k}\frac{k}{\rho}\frac{1}{\omega-n\Omega}e^{ikx+kz}\right). \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (7), we obtain $\nu_{x,n,k}=(iq_n/k)\nu_{z,n,k}$ and

$$\nu_{1x}=e^{-i\omega t}\sum_{n=-\infty}^{\infty}e^{in\Omega t}\left(i\frac{q_n}{k}\nu_{z,n,k}e^{ikx+q_n z}+p_{n,k}\frac{k}{\rho}\frac{1}{\omega-n\Omega}e^{ikx+kz}\right). \tag{18}$$

Now we need to take boundary conditions (4) and (5) into account. Presuming that the perturbation amplitude is small compared to the wavelength, we may put $z=0$ into the boundary condition [Eq. (5)] and finally obtain

$$p_{n,k}=i\frac{(n\Omega-\omega)\left(2+i\dfrac{n\Omega-\omega}{\nu k^2}\right)}{2k}\rho\nu_{z,n,k}. \tag{19}$$

Condition (4), with the same assumption of small amplitude, can be rewritten in the linear approximation as follows:

$$-p_0|_{z=\zeta}-p_1|_{z=0}+2\eta\frac{\partial \nu_{1z}}{\partial z}\bigg|_{z=0}=-p_0|_{z=0}. \tag{20}$$

Here ζ is the displacement of the particles along the z axis. Making use of Eq. (19) we can rewrite Eq. (20):

$$0=-e^{-i\omega t}\sum_{n=-\infty}^{\infty} e^{in\Omega t} i\,\frac{(n\Omega-\omega)\left(2+i\,\dfrac{n\Omega-\omega}{\nu k^2}\right)}{2k}\,\rho \nu_{z,n,k} e^{ikx}$$
$$+2\eta e^{-i\omega t}\sum_{n=-\infty}^{\infty} e^{in\Omega t} q_n \nu_{z,n,k} e^{ikx}$$
$$-2\eta i e^{-i\omega t}\sum_{n=-\infty}^{\infty} e^{in\Omega t} i\,\frac{(n\Omega-\omega)\left(2+i\,\dfrac{n\Omega-\omega}{\nu k^2}\right)}{2k}$$
$$\times \rho \nu_{z,n,k}\frac{k^2}{\rho}\,\frac{1}{\omega-n\Omega}\,e^{ikx}-\rho[g+\xi_0\Omega^2\cos(\Omega t)]\zeta. \quad (21)$$

Now we should take into account that $\nu_{1z}=d\zeta/dt\approx\partial\zeta/\partial t$, and, in the vicinity of the surface $z=0$,

$$\nu_{1z}=e^{-i\omega t}\sum_{n=-\infty}^{\infty} e^{in\Omega t} e^{ikx} i\rho\,\frac{\omega-n\Omega}{2\eta k^2}\,\nu_{z,n,k}, \quad (22)$$

$$\zeta=-e^{-i\omega t}\sum_{n=-\infty}^{+\infty} e^{in\Omega t} e^{ikx}\rho\,\frac{\nu_{z,n,k}}{2\eta k^2}$$
$$=e^{-i\omega t}\sum_{n=-\infty}^{+\infty} e^{in\Omega t} e^{ikx}\zeta_{n,k},$$

$$\nu_{z,n,k}=-2\,\frac{\eta}{\rho}k^2\zeta_{n,k}. \quad (23)$$

Substituting Eqs. (22) and (23) into Eq. (21), we finally obtain

$$\sum_{n=-\infty}^{+\infty}\left[\left(2+i\rho\,\frac{n\Omega-\omega}{\eta k^2}\right)^2 -4\left(1+i\rho\,\frac{n\Omega-\omega}{\eta k^2}\right)^{1/2}-\frac{g\rho^2}{\eta^2 k^3}\right]\zeta_{n,k}e^{in\Omega t}$$
$$=\sum_{n=-\infty}^{+\infty} e^{in\Omega t}\frac{\xi_0\Omega^2\rho^2}{\eta^2 k^3}\cos(\Omega t)\zeta_{n,k}. \quad (24)$$

Designating

$$F(\omega-n\Omega)=\left(2+i\rho\,\frac{n\Omega-\omega}{\eta k^2}\right)^2-4\left(1+i\rho\,\frac{n\Omega-\omega}{\eta k^2}\right)^{1/2}-\frac{g\rho^2}{\eta^2 k^3}, \quad (25)$$

we can rewrite Eq. (24) as

$$\sum_{n=-\infty}^{+\infty} e^{in\Omega t}F(\omega-n\Omega)\zeta_{n,k} -\sum_{n=-\infty}^{+\infty} e^{in\Omega t}\frac{\xi_0\Omega^2\rho^2}{\eta^2 k^3}\,\frac{e^{i\Omega t}+e^{-i\Omega t}}{2}\,\zeta_{n,k}=0. \quad (26)$$

Reindexing the subscripts, we obtain

$$\sum_{n=-\infty}^{+\infty} e^{in\Omega t}\left(F(\omega-n\Omega)\zeta_{n,k}-\frac{\xi_0\Omega^2\rho^2}{2\eta^2 k^3}(\zeta_{n-1,k}+\zeta_{n+1,k})\right)=0. \quad (27)$$

Condition (27) is certainly satisfied if, for all $n=(-\infty, +\infty)$,

$$\psi\zeta_{n-1,k}+F(\omega-n\Omega)\zeta_{n-1,k}+\psi\zeta_{n+1,k}=0, \quad (28)$$

$$\text{where}\quad \psi\equiv-\frac{\xi_0\Omega^2\rho^2}{2\eta^2 k^3}. \quad (29)$$

The system of algebraic equations (28) has a nontrivial solution $\zeta_{n,k}$ if

$$\mathrm{Det}\begin{pmatrix} . & . & . & . & . & . & . \\ \psi & F(\omega-2\Omega) & \psi & 0 & 0 & 0 & 0 \\ 0 & \psi & F(\omega-\Omega) & \psi & 0 & 0 & 0 \\ 0 & 0 & \psi & F(\omega) & \psi & 0 & 0 \\ 0 & 0 & 0 & \psi & F(\omega+\Omega) & \psi & 0 \\ 0 & 0 & 0 & 0 & \psi & F(\omega+2\Omega) & \psi \\ . & . & . & . & . & . & . \end{pmatrix}=0 \quad (30)$$

Therefore, the sought-after dispersion equation in our problem is presented in the form of an infinite determinant [Eq. (30)]. Up to this moment our consideration was similar to the research of Ref. [5]. However, in our further analysis let us proceed by further transforming the infinite determinant [Eq. (30)] and extracting the variable ω to a simple trigonometric function. This would allow one to avoid a numerical inversion of the matrix performed in Ref. [5] for solving the eigenvalue problem.

III. HIGH FREQUENCY–SMALL AMPLITUDE EXTERNAL OSCILLATIONS

The form [Eq. (30)] of the dispersion equations is already convenient for considering high frequency forced external oscillations, $\Omega \gg |\omega|$. For further approximations we will also use dimensionless frequencies and characteristic scales, namely, $\tilde{\omega} \equiv \omega/\omega_\nu$, $\tilde{k} \equiv k/k_\nu$, $\tilde{\Omega} \equiv \Omega/\omega_\nu$, and $\tilde{\xi}_0 \equiv \xi_0 k_\nu$, where

$$\omega_\nu \equiv \left(\frac{g^2\rho}{\eta}\right)^{1/3}, \quad k_\nu \equiv \left(\frac{g\rho^2}{\eta^2}\right)^{1/3}. \tag{31}$$

Under the above-mentioned condition of high Ω, one can rewrite Eq. (25) (for $n \neq 0$):

$$F(\omega - n\Omega) \approx \left(2 + i\frac{n\tilde{\Omega}}{\tilde{k}^2}\right)^2 - 4\left(1 + i\frac{n\tilde{\Omega}}{\tilde{k}^2}\right)^{1/2} - \frac{1}{\tilde{k}^3}, \tag{32}$$

and also

$$\psi \equiv -\frac{\tilde{\xi}_0\tilde{\Omega}^2}{2\tilde{k}^3}. \tag{33}$$

For small amplitudes, $\tilde{\xi}_0\tilde{k} \ll 1$, we may compare $|F(\omega - n\Omega)|$ and $|\psi|$ at large ($\tilde{k}^2 \ll \tilde{\Omega}$) and short ($\tilde{k}^2 \gg \tilde{\Omega}$) wavelengths of the perturbations. In both cases we can satisfy $|F(\omega - n\Omega)| \gg |\psi|$, $\forall n \neq 0$, and may use only three central rows of the determinant (30):

$$\mathrm{Det}\begin{pmatrix} F(\omega-\Omega) & \psi & 0 \\ \psi & F(\omega) & \psi \\ 0 & \psi & F(\omega+\Omega) \end{pmatrix} = F(\omega-\Omega)F(\omega+\Omega) \times \left[F(\omega) - \psi^2\left(\frac{1}{F(\omega+\Omega)} + \frac{1}{F(\omega-\Omega)}\right)\right] = 0. \tag{34}$$

For $\tilde{k}^2 \ll \tilde{\Omega}$, Eq. (34) can be further simplified since we have, from Eq. (32),

$$F(\omega \pm \Omega) \approx F(\pm\Omega) \approx -\frac{\tilde{\Omega}^2}{\tilde{k}^4}. \tag{35}$$

Designating $\gamma \equiv -i\tilde{\omega}$ and using Eqs. (34) and (35) together, we can find a final dispersion equation of the form

$$\left(2 + \frac{\gamma}{\tilde{k}^2}\right)^2 - 4\left(1 + \frac{\gamma}{\tilde{k}^2}\right)^{1/2} - \frac{1}{\tilde{k}^3} + \frac{1}{2}\left(\frac{\tilde{\xi}_0\tilde{\Omega}}{\tilde{k}}\right)^2 = 0. \tag{36}$$

One can find an asymptotic expression of Eq. (36) for very high wavelengths of the perturbations, $\tilde{k}^2 \ll \gamma$ (the same approximation appears in the case of low viscosity, $\eta \to 0$, as it should be):

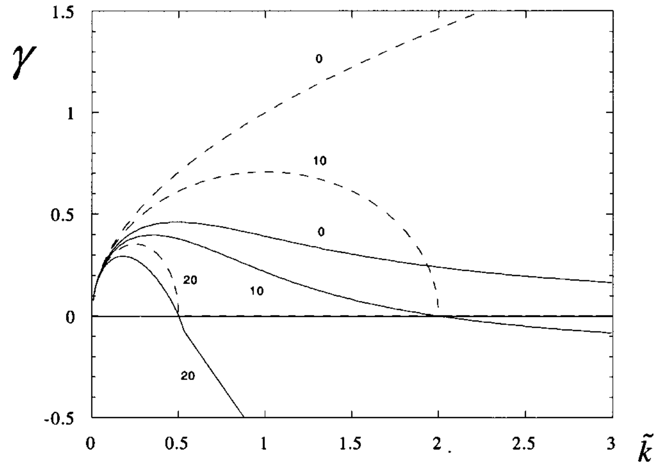

FIG. 2. Example of the spectrum $\gamma(\tilde{k})$ in the small amplitude–high frequency limit for small wave numbers. Here $\tilde{\xi}_0 = 0.1$ [$(\xi_0 k)_{max} = 0.3$]. Equation (36) is used for solid lines, and asymptote (37) for dashed ones. Numbers correspond to different values of $\tilde{\Omega}$.

$$\gamma \approx \sqrt{\tilde{k}\left(1 - \frac{\tilde{k}}{2}\tilde{\xi}_0^2\tilde{\Omega}^2\right)^{1/2}}. \tag{37}$$

Therefore, externally imposed oscillations do reduce the instability growth rate below the classical RT value $\gamma = \sqrt{\tilde{k}}$. However, as we consider $\tilde{\xi}_0\tilde{k} \ll 1$, a remarkable instability attenuation can be achieved only for rather high amplitudes of the acceleration field, $|\mathbf{g}_{osc}|_{max} \equiv \tilde{\xi}_0\tilde{\Omega}^2 g \gg g$.

This situation is illustrated by Fig. 2. The function $\gamma(\tilde{k})$ [that is, the maximum Re of solutions of Eq. (36)] is presented for $\tilde{\xi}_0 = 0.1$ and several values of $\tilde{\Omega}$. Starting from some $\tilde{k}$ ($\tilde{k} = 1$ for $\Omega = 0$) we have an additional negative root ($\gamma < -0.5$, not shown), which does not represent instability. Also is plotted the asymptote dependence [Eq. (37)] (for $\tilde{\Omega} = 0$ it corresponds to the classical value in incompressible fluid).

In the same manner, for the limit of short wavelengths, $\tilde{k}^2 \gg \tilde{\Omega}$, we have the following asymptote from Eqs. (32) and (34),

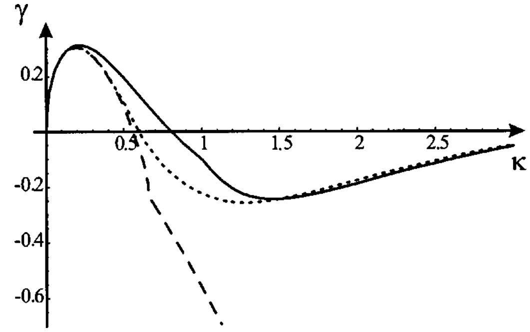

FIG. 3. Exact solution $\gamma(\tilde{k})$ (solid curve) and the approximate solutions (34) (dotted curve), and (36) (dashed curve); $\tilde{\xi}_0 \approx 0.18$ and $\tilde{\Omega} = 10.6$.

$$\gamma \approx \frac{1}{2\tilde{k}}\left(1-\frac{3}{16}\frac{\tilde{\xi}_0^2\tilde{\Omega}^4}{\tilde{k}^3}-\frac{3}{8}\frac{1}{\tilde{k}^3}\right), \tag{38}$$

and see again the effect of the instability attenuation with respect to the classical viscous spectrum, $\gamma = 1/2k$, at least for $\tilde{\xi}_0\tilde{\Omega}^2 > 1$. Finally, in order to examine the whole area of ξ_0, k, and Ω, we need to come back to the full infinite determinant [Eq. (30)].

IV. EXACT SOLUTION OF THE DISPERSION EQUATION

In order to solve Eq. (30), we will use the method developed in Refs. [6,7], and based on the method of Hill [8]. First of all, let us draw our attention to the fact that the equation $F(\omega_0)=0$ [see Eq. (25) and Fig. 2] gives the RT instability in viscous fluid [4]. If we impose additional forced oscillations, we will not have the same solution, and hence $F(\omega_{\text{solution}})\neq 0$. In a similar way we can presume that $\omega \neq n\Omega + \omega_0$ (except maybe for some particular values of k). Thus we can divide each row of the determinant [Eq. (30)] by the corresponding diagonal term $F(\omega - n\Omega)$:

$$D(\omega)=\text{Det}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\psi}{F(\omega-2\Omega)} & 1 & \frac{\psi}{F(\omega-2\Omega)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\psi}{F(\omega-\Omega)} & 1 & \frac{\psi}{F(\omega-\Omega)} & 0 & 0 & 0 \\ 0 & 0 & \frac{\psi}{F(\omega)} & 1 & \frac{\psi}{F(\omega)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\psi}{F(\omega+\Omega)} & 1 & \frac{\psi}{F(\psi+\Omega)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\psi}{F(\omega+2\Omega)} & 1 & \frac{\psi}{F(\omega+2\Omega)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{39}$$

Since the diagonal terms are all equal to unity, and the adjacent to diagonal terms are decreasing in value as $1/n^2$ [in accordance with Eq. (25)], the determinant is converging.

Let us now consider determinant (39) as a function of complex variable ω. Since the determinant converges, this function is analytic everywhere in the complex area, but the nulls of the functions $F(\omega - n\Omega)=0$. In this case the poles of $D(\omega)$ originate from the terms $1/F(\omega - n\Omega)$, and in order to find the main part of $D(\omega)$ we need to sum up first over all the roots ω_0 of the equation $F(\omega_0)=0$ and then over all the rows of the matrix. Thus

$$D(\omega)=\varphi(\omega)+\sum_{\omega_0}\sum_{n=-\infty}^{+\infty}\frac{\psi}{\left.\frac{\partial F(\omega)}{\partial\omega}\right|_{\omega=\omega_0}[\omega-(\omega_0-n\Omega)]}D_1(\omega_0), \tag{40}$$

where $\varphi(\omega)$ is an integral function and $D_1(\omega_0)$ is the determinant of the matrix of Eq. (39) with a regularized row containing a peculiarity at $\omega=\omega_0$:

$$D_1(\omega_0)=\text{Det}\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\psi}{F(\omega_0-2\Omega)} & 1 & \frac{\psi}{F(\omega_0-2\Omega)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\psi}{F(\omega_0-\Omega)} & 1 & \frac{\psi}{F(\omega_0-\Omega)} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{\psi}{F(\omega_0+\Omega)} & 1 & \frac{\psi}{F(\omega_0+\Omega)} & 0 \\ 0 & 0 & 0 & 0 & \frac{\psi}{F(\omega_0+2\Omega)} & 1 & \frac{\psi}{F(\omega_0+2\Omega)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{41}$$

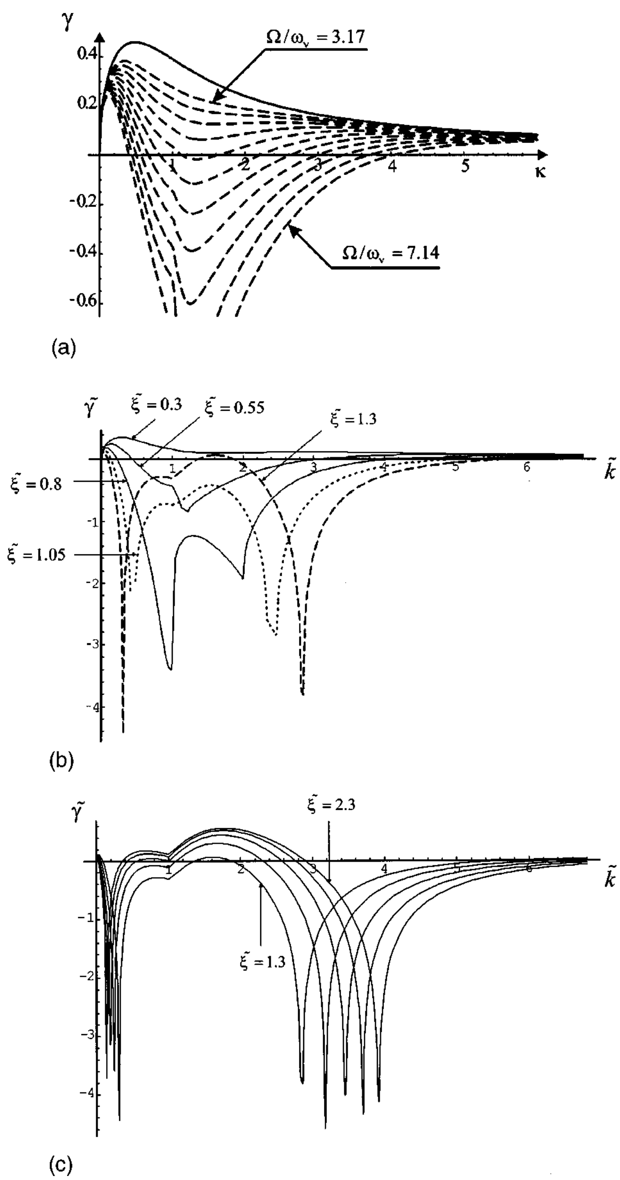

FIG. 4. Solutions $\gamma(\tilde{k})$ of (43): (a) $\xi_0=0.4$; $\tilde{\Omega}$ varies from 3.17 to 7.14 with the step of 0.4. (b) $\tilde{\Omega}=5$; ξ_0 varies from 0.3 to 1.3. (c) $\tilde{\Omega}=5$; ξ_0 varies from 1.3 to 2.3.

The function $\varphi(\omega)$ tends to unity at $\omega\to\infty$ [since $D(\omega)\to 1$ at $\omega\to\infty$]. Due to the Liouville theorem, this function is equal to unity in all the complex area. Then, reminding ourselves of the Mittag-Leffler decomposition of meromorphic functions to series, we may represent Eq. (40) in the form

$$D(\omega)=1+\sum_{\omega_0}\left.\frac{\psi}{\partial F(\omega)/\partial\omega}\right|_{\omega=\omega_0}\frac{\pi}{\Omega}\cot\left(\frac{\pi(\omega-\omega_0)}{\Omega}\right)D_1(\omega_0)=0. \tag{42}$$

As we can see, the frequency ω we are looking for is placed now as an argument of a simple trigonometric function: the cotangent. The determinant $D_1(\omega_0)$ does not depend on the solution $\omega(k)$, and rapidly converges. This means that for a given k we can easily find the corresponding growth rate.

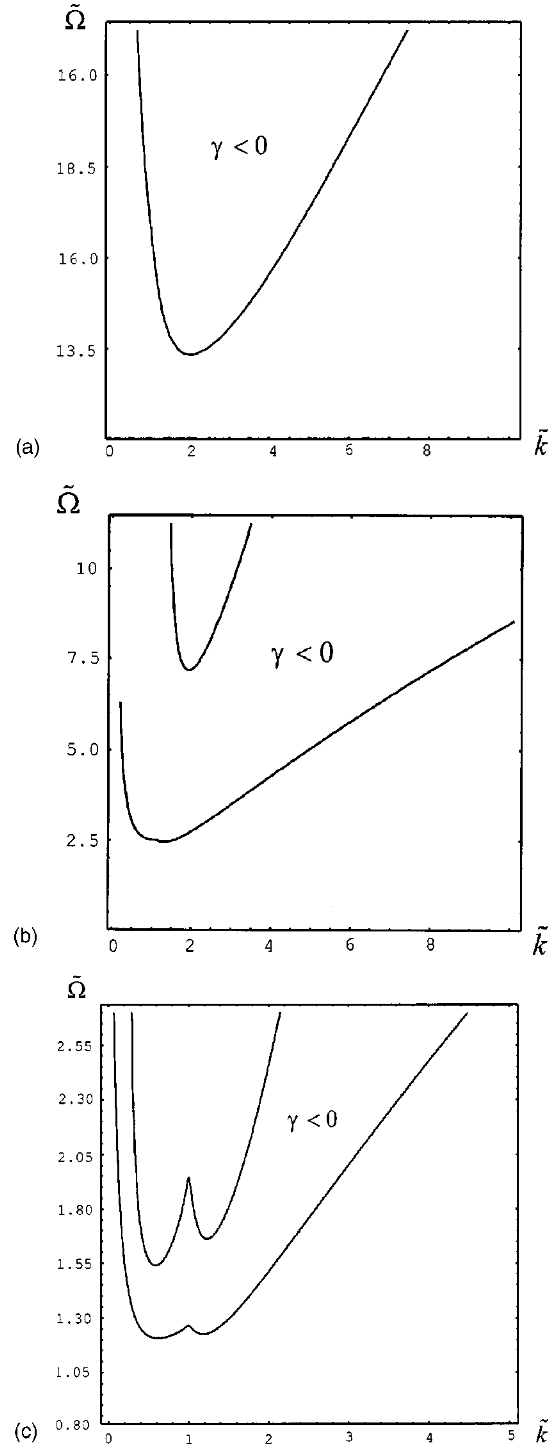

FIG. 5. Regions of stability at different wave numbers for small, intermediate, and high ξ_0: (a) $\tilde{\xi}_0=0.1$. (b) $\tilde{\xi}_0=1$. (c) $\tilde{\xi}_0=3.0$.

Using Eq. (31) we will calculate the instability spectrum from resulting dispersion relation [Eq. (42)] in a dimensionless form:

$$1+\frac{\pi\psi}{\tilde{\Omega}}\sum_{\gamma_0}\left.\frac{1}{\partial F(\gamma)/\partial\gamma}\right|_{\gamma=\gamma_0}\mathrm{cth}\left(\pi\frac{\gamma-\gamma_0}{\tilde{\Omega}}\right)D_1(\gamma_0)=0, \tag{43}$$

where

$$F(\gamma)=\left(2+\frac{\gamma}{\tilde{k}^2}\right)^2-4\left(1+\frac{\gamma}{\tilde{k}^2}\right)^{1/2}-\frac{1}{\tilde{k}^3}, \tag{44}$$

and ψ comes from Eq. (33).

Again, as in Sec. III, the exact solution [Eq. (43)] takes simpler form for high frequency external oscillations, i.e., $\tilde{\Omega}\gg|\gamma|$, $|\gamma_0|$, and hence $\tilde{\Omega}\gg|\gamma-\gamma_0|$. Only three central rows of $D_1(\gamma_0)$ contribute when $\tilde{k}^2\ll\tilde{\Omega}$ or $\tilde{k}^2\gg\tilde{\Omega}$, since in these cases $|F(\omega_0-n\Omega)|\gg|\psi|$. Therefore,

$$\gamma-\gamma_0-\psi^2\sum_{\gamma_0}\left.\frac{1}{\partial F(\gamma)/\partial\gamma}\right|_{\gamma_0}\left(\frac{1}{F(\gamma_0+\Omega)}+\frac{1}{F(\gamma_0-\Omega)}\right)=0. \tag{45}$$

We recall that the sum in Eq. (45) should be calculated for the roots of equation $F(\gamma_0)=0$. The second root appears only for $\tilde{k}\geq 1$ and is equal to $\gamma_0\approx-0.912\tilde{k}^2$. It can be easily seen that for this root $\partial F/\partial\gamma|_{\gamma_0}\propto 1/\tilde{k}^2$, while for the first root and $\tilde{k}\gg\gamma_0$ we have $\partial F/\partial\gamma|_{\gamma_0}\propto\gamma_0/\tilde{k}^4$. Thus for $\tilde{k}<1$ and $\tilde{k}\gg 1$ one can consider only the first (positive) root γ_0 in Eq. (45), which has asymptotes $\gamma_0\rightarrow\sqrt{\tilde{k}}$ and $\gamma_0\rightarrow 1/2\tilde{k}$ for these two cases correspondingly.

For small deviations of the growth rate from the value of $\gamma_0>0$, such that one may write $F(\gamma)\approx\partial F/\partial\gamma|_{\gamma_0}(\gamma-\gamma_0)$, Eq. (45) immediately coincides with Eq. (34). In addition, for arbitrary $(\gamma-\gamma_0)/\gamma_0$, Eq. (45) can be approximated for $\tilde{k}^2\ll\tilde{\Omega}$ or $\tilde{k}^2\gg\tilde{\Omega}$, and exactly repeats asymptotes (37) and (38). Comparison of the exact solution with previously found approximations is presented in Fig. 3. Good approximation of the exact solution appears in the area of small and large $\tilde{k}$, in full agreement with the discussion of Sec. III.

Let us now examine the behavior of the exact spectrum $\gamma(\tilde{k})$, resulting from Eq. (43). A gradual increase of the frequency causes the dispersion curve to plunge to an area of negative values of the growth rate, making the window of stability larger and larger, [Fig. 4(a)]. However, if $\tilde{\Omega}$ is too large, additional peaks tend to appear on the dispersion curve. The value of γ in the area of these peaks can exceed the growth rate of the original instability with $\Omega=0$, and hence the parametric oscillations even cause an antidamping of the perturbations. The influence of the dimensionless amplitude parameter $\tilde{\xi}_0$ is in certain way similar—if it is large enough, the external oscillations become a destabilizing factor in some region of wave numbers [Figs. 4(b) and 4(c)]. Finally, the exact solution [Eq. (43)] allows one to draw the areas of stability in the form of Fig. 5.

V. LIMIT CASE OF IDEAL INCOMPRESSIBLE FLUID

An interesting limit case of the problem is the transition to the case of zero viscosity, i.e., $\eta\rightarrow 0$. This transition, while keeping Ω/ω_ν constant, corresponds to $\Omega\rightarrow\infty$. It is in fact the case of an ideal incompressible fluid in a gravitational field, i.e., the case of classic Rayleigh-Taylor instability. The only difference is that we consider it in the oscillating gravity field. The problem was investigated previously, starting in Ref. [9]. We are using it as a demonstration of our approach to transforming the determinant. One can apply the same technique of regularization of an infinite determinant as for the problem of a viscous fluid. The result can be obtained in a form close to Eq. (42),

$$\sin^2\left(\pi\frac{\omega}{\Omega}\right)-\sin^2\left(\pi\frac{i\gamma_0}{\Omega}\right)-\frac{\pi}{4}\psi\frac{i\gamma_0}{\Omega}\sin\left(2\pi\frac{i\gamma_0}{\Omega}\right)D_1(i\gamma_0)=0, \tag{46}$$

where

$$D_1(i\gamma_0)=\mathrm{Det}\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\frac{\psi}{F(i\gamma_0-2\Omega)}&1&\frac{\psi}{F(i\gamma_0-2\Omega)}&0&0&0&0\\
0&\frac{\psi}{F(i\gamma_0-2\Omega)}&1&\frac{\psi}{F(i\gamma_0-\Omega)}&0&0&0\\
0&0&1&0&1&0&0\\
0&0&0&\frac{\psi}{F(i\gamma_0+\Omega)}&1&\frac{\psi}{F(i\gamma_0+\Omega)}&0\\
0&0&0&0&\frac{\psi}{F(i\gamma_0+2\Omega)}&1&\frac{\psi}{F(i\gamma_0+2\Omega)}\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}, \tag{47}$$

and also

$$\gamma_0 = \sqrt{kg} \quad F(\omega) = 1 + \frac{\omega^2}{kg}, \quad \psi = \frac{\xi_0 \Omega^2}{g}.$$

The corresponding dispersion curve for $\xi_0\Omega^2/g > 1$ is drawn in Fig. 6. At small wave numbers the growth rate is smaller that of the ''pure'' RT instability γ_0. Then its imaginary part is equal to zero, and finally (at large k) becomes higher than γ_0. We note that this result can also be reformulated for the Kapitsa pendulum [2]. The problem of the pendulum will be solved for arbitrary values of the attachment point oscillation amplitudes, that are not small compared to the length of the pendulum.

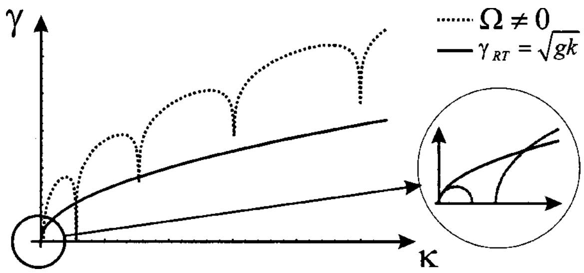

FIG. 6. Dispersion curve for an ideal incompressible fluid imposed on external oscillations. The inset zooms in on the region of small wave numbers.

VI. CONCLUSION

In conclusion, the stability of a viscous fluid interface is analyzed for the case of an oscillating gravitational field. A dispersion relation is derived in the linear approximation to represent an infinite determinant of the Hill type. The applied method of regularization of this determinant allows one to find the dispersion curve, and to determine the stability regions for a broad range of parameters. It is also shown that in the limit of inviscid fluid (in analogy with the Kapitsa pendulum) the external oscillations lead to a resonant antidamping of short wavelength perturbations.

Thus the phenomenon of boundary surface oscillation can potentially be used for the instability suppression in a certain region of wavelengths. However, one should choose the parameters of this ''pumping'' carefully, so that no inverse destabilization effect occurs. A practical realization of this method in liquids is obvious, and the effect was already experimentally demonstrated (see, e.g., Refs. [3,5]). One may imagine a similar effect in magnetically accelerated plasmas with a frozen-in magnetic field $B_{\rm in}$. If we consider a sharp boundary (neglecting the effect of the magnetic field diffusion) and an external accelerating magnetic pressure having oscillations, the sum force could entail the desired variation of the acceleration, $g \propto B^2_{\rm ext}(t) - B^2_{\rm in}$.

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